Vibration of Plates
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Preface to the Reprinted Edition

The Acoustical Society of America is reprinting this book, along with my subsequently published one, "Vibration of Shells," as a service to the technical community with no commercial expectations. I am very pleased that they are doing this, for both books have been in great demand, and have been out of print for a long time.

For example, 5100 copies of "Vibration of Plates" were published by the U.S. Government Printing Office in 1969. 100 copies were hard bound, and were given to key people in the technical community. Of the remaining 5000 soft-bound copies, approximately 1000 were distributed to NASA and other government personnel and technical libraries, and the remainder were made available inexpensively to the general public by the Superintendent of Documents. Within three years they were all bought. Hearing this, in 1974 I approached the NASA publication personnel in Washington to get GPO to reprint the book. They were told by GPO that it would not be reprinted because of "insufficient demand!" Such is bureaucratic logic. Being at a low point in my life at that time, and not willing to fight the battle myself to get the book reprinted, I let the matter drop.

I wrote this book because I felt that there was a great need for it. At least I needed it, and therefore thought that many others did, too. In this respect my expectations have been exceeded. During the intervening quarter century since it appeared, I have received scores of letters and phone calls from people, often asking questions, who are grateful for its existence. A study of Science Citation Index in 1987 showed that during the preceding seven years, "Vibration of Plates" was cited by others 226 times in the published literature--an average of 32 times per year. Thus, I feel that my time was well spent in writing this book.

Although this book was first published in 1969, its writing was finished in 1966. In the original Preface, reasonable completeness of results published in the scientific and technical literature through 1965 was claimed. Conversely, only a few last-minute additions were made of results published in 1966. Subsequently, I have uncovered only a few significant references which are not included among the 500 in this work, although they should have been. So, I am satisfied with its "reasonable completeness."

However, an enormous number of publications dealing with free vibrations of plates have appeared since 1965. I have 1500 additional ones in my file cabinets! The upsurge in research and writing on the subject began several years after the book first appeared, when many researchers realized what specific cases have received little or no attention, and continues unabated today. Most of this later research deals with what I call "complicating effects," beyond what is governed by classical, isotropic, thin plate theory. These complicating effects are anisotropy (Chapter 9), inplane forces (10), variable thickness (11), and other considerations (12)
such as the effects of surrounding media (e.g., water, air), large (nonlinear) deflections, shear deformation and rotary inertia, and nonhomogeneity. The largest increase in recent publications is in the topic area of nonhomogeneity, where hundreds of papers examining laminated composite plates have appeared.

It was stated in the original Preface that I intended to "write a supplement to this volume after a few years have elapsed." Indeed, this is why I have continued to collect relevant references. So far, I have not been able to make the time available to do so, but I still intend to. Certainly with all the new research results appearing since 1965, such a work is greatly needed. But it will have to be more than a "supplement." I foresee a complete rewriting, with frequencies, frequency determinants, eigenfunctions, nodal patterns, and mode shapes, summarized in four volumes. Of course, this very ambitious project will require much time. In the meantime I would continue to appreciate receiving relevant information from researchers, especially that not easily available in the well known journals (e.g., reports, theses, conference papers).


As could be expected, some errors did appear in this work. I discovered some while using the book hundreds of times in the past 24 years. Others were pointed out by researchers in letters and phone calls. A total of sixteen errors were found that have been corrected in this reprinted edition. Corrections are on pages, 5, 10, 11, 14, 29, 53, 60, 67, 77, 105, 144, 262, 277, and 285.

I want to thank Dr. Mauro Pierucci of San Diego State University who, as Chairman of the Book Committee of the Acoustical Society of America, initiated this reprinting effort and followed it through to its fruition.

Arthur W. Leissa
The Ohio State University
May 10, 1993

Preface to the Original Edition

The ever-increasing rate of scientific research throughout the world, and particularly in the United States, is a well-known fact. This increase is partly evidenced by the growing number of books, papers, and reports published every year. Indeed, we are faced with an information retrieval problem. If the results of a piece of scientific work are to provide useful knowledge, the expository technical papers or reports must be generally known and available, and they must be capable of being understood and evaluated by the reader (the problem of language is included here). The present monograph attempts to bridge these gaps in one field—the vibration of plates.

From the beginning, two objectives were intrinsic in this work:

(1) A comprehensive set of available results for the frequencies and mode shapes of free vibration of plates would be provided for the design or development engineer.

(2) A summary of all known results would be provided for the researcher in the field of plate vibrations.

These objectives will be elaborated upon below.

Several years ago I observed the following incident at a major aerospace company. An engineer needed to know the first three frequencies and mode shapes of a rectangular plate of a certain aspect ratio and with certain simple restraint conditions along its edges. A literature search was conducted by the engineer for 2
weeks, during which only the first two frequencies and no accurate mode shapes were found. Since he had
neither the analytical capability of solving the problem nor the money and time needed for an experimental
program, the engineer was forced to drop the problem at this point.

In the present study all direct results which are known for the aforementioned problem are presented.
Furthermore, from a brief comparison among the known results for other boundary conditions, estimates of
additional frequencies and mode shapes can be made. This is one way in which the engineer can develop a
qualitative understanding of plate vibrational behavior. For the aforementioned problem, at least two
approximate formulas are given for estimates of frequencies. Finally, the mathematical techniques used in the
literature to solve the problem or related ones are pointed out in case more accurate results are needed.

It is my hope that this monograph will reduce duplication of research effort in plate vibrations in the future (a
very pointed example is that of the square plate clamped all around). In addition, the researcher is provided
accurate numerical results for the testing of new methods (this is the reason that results are given to eight
significant figures in some cases). Finally, it is hoped that this work will give added perspective to the merits
and complexities of applying analytical techniques to eigenvalue problems.

Gaps in knowledge are made implicitly obvious by examining this work. For example, analytical results have
been found for a clamped elliptical plate, and experimental results for the free case, but no results whatsoever
have been found for the simply supported case.

The scope of this study was limited by several considerations. Only the analytical results from plate theories
were considered; that is, the governing equations are two-dimensional, not three-dimensional. Materials were
restricted to those which are linearly elastic. Structures were not included in the study; for example, a
rectangular plate supported by one or more edge beams was considered to be a structure.

The primary logical division of this work is by the complexity of the governing differential equations. Thus, the
first eight chapters deal with the simplest "classical theory" of plates. The next three chapters introduce the
complications of anisotropy, in-plane force, and variable thickness. Other complications are discussed in the
twelfth chapter. The first subdivision is by geometrical shape; that is, circles, ellipses, rectangles,
parallelograms, and so forth. Further subdivision accounts for holes, boundary conditions, added masses or
springs, and so forth.

It is presupposed that the user of this monograph will have at least an elementary understanding of plate
theory. In order to increase understanding and to define notation and assumptions more clearly, a reasonably
rigorous derivation of the plate equations is made in the appendix.

Some statements about the format of presentation will be useful in understanding this work. It will be seen that
the majority of results available are for the natural frequencies of free vibration and quite often only the
fundamental (lowest) frequency. Patterns showing node lines are frequently available for the higher modes.
Mode shapes (deflection surfaces in two dimensions) are usually not completely specified in the literature. It
should be remarked here that the mode shapes (eigenfunctions) cannot be completely determined until the
frequencies (eigenvalues) are found. The mode shapes are generally known less accurately than the
frequencies.

Virtually no one in the literature evaluates the bending stresses due to a unit amplitude of motion. This
information is obviously important, particularly for fatigue studies. The lack of results is undoubtedly due to the
fact that the stresses must be obtained from second derivatives of the mode shapes. Not only does this require
additional computational work, but also the mode shapes usually are not known with sufficient accuracy to give
meaningful results for stresses.

Frequency data were converted to the angular frequency \( w \) (radians/unit time) or to a corresponding
nondimensional frequency parameter, where possible. Almost always the number of significant figures was
kept the same as that in the original publications. In no case were significant figures added. In some few cases
the number of significant figures was reduced because the accuracy of the calculations in the publication did
not justify the numbers given. Curves were not replotted but were photographically enlarged and traced to
maximize accuracy. Quite often, when they are available, both tabular and graphical results are given for a
problem. Tabular results are particularly important for measuring the accuracy of an analytical method,
whereas curves are valuable for interpolation, extrapolation, and qualitative studies. In some cases many sets
of results are given for the same problem. In these cases each set was derived by a different theoretical or
experimental technique; this permits a comparison of techniques.

Two of the major goals of the project were accuracy and completeness. Some of the efforts made to maintain
accuracy have been described in the foregoing paragraphs. Reasonable completeness of results published
through the end of the year 1965 is claimed. Writing of the manuscript began in the summer of that year. In
addition to the well-known abstracting journals, several special-purpose bibliographies were used in order to
procure pertinent technical papers and reports. Further references were obtained from the discussion and
reference lists within those already procured. Approximately 150 letters were sent to people throughout the
world who were known to be active in the field of continuum vibrations. These letters listed their publications
already in hand and asked for copies of any others which they deemed applicable. Through these efforts I
have come to possess a reasonably complete set of literature in the field of plate vibrations. However, in spite
of this, I am convinced that some significant publications are not included, particularly some which are know
to exist but have been thus far unobtainable, especially books by Soviet researchers.

In light of the preceding paragraph, I expect—indeed, hope—to receive considerable valuable criticism pointing
out errors or omissions. In addition, I would appreciate receiving copies of recent or forthcoming publications
and reports which are pertinent. It is my intention to write a supplement to this volume after a few years have
elapsed; such a document will correct any major mistakes or omissions in this work and will report on further
advances in the field.

For historical record and recognition it should be pointed out that, approximately 6 months after this project
began, I discovered a notable work entitled "Free Vibrations of Plates and Shells," by V.S. Gontkevich,
published (in Russian) in 1964. A subsequent complete translation into English was made under the
sponsorship of the Lockheed Missiles & Space Co. This book purports to do what the present monograph does
and, in addition to plates and shells, covers the fields of membranes and stiffened plates. I do not wish to
criticize the work of Mr. Gontkevich. Indeed, if used with great care, his work can be used to supplement this
monograph. Nevertheless, two objective comments concerning Gontkevich's work must be made for the
record:

(1) The number of references on plate vibrations included is less than half of those in the present monograph.

(2) the large number of typographical mistakes made and the difficulty in interpreting the work (in either the
original Russian or in the English translation) decrease its usefulness enormously.

The present monograph, sponsored by the National Aeronautics and Space Administration, is my first major
undertaking in the area of continuum vibrations. It is to be continued by a 2-year project which is currently in
progress and summarizes the field of vibrations of shells. I would appreciate receiving technical papers and
reports related to that field from the readers of this work.

The support of the National Aeronautics and Space Administration is gratefully acknowledged. In particular, I
am indebted to Mr. Douglas Michel of NASA, who not only recognized the potential value of this work, but was
thinking of it before my proposal ever reached him. His technical comments and advice during the course of
the work were also greatly appreciated. I particularly wish to thank Messrs. Milton Vagins and S.G. Sampath,
who did all the necessary work so that I could be free for the actual summarization and writing. Without their
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